On absence of relativistic damping of electron Bernstein waves in tokamak plasmas

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1 Introduction

Electron Bernstein waves (EBW) rays calculations show invariably that only the waves launched close to the mid-plane are capable of travelling deep into the plasma volume. This effect has an obvious explanation. The waves are produced in the upper hybrid resonance (UHR) with a rather small wave - vector growing fast in the inhomogeneous medium as the wave leaves the UHR vicinity. Far from the mid - plane the plasma inhomogeneity is essentially two-dimensional and there is no reason for the wave-vector to grow perpendicular to the magnetic field. As a result, a large wave -vector projection on the magnetic field appears leading to fast wave damping. By contrast, in the mid-plane region the inhomogeneity is almost one-dimensional and the wave-vector grows nearly perpendicular to the magnetic field. Since deep wave penetration is of interest for EBW plasma heating and ECE diagnostics, in this report we analyze the wave behavior close to the mid-plane.

2 Analytical treatment of the EBW rays close to ECR layer

The form of the EBW rays depends of the shape of the ECR surface. In the case of concave surface the rays oscillate regularly around the mid-plane, while for the convex surface rays deviation off the mid-plane grows exponentially (fig.1). In this work the first case is considered. Regular behavior of EBW rays makes possible their analytical description. Consider a domain within the plasma volume located symmetrically relative to the tokamak mid-plane. Introduce the Cartesian co-ordinate system (x, y, z) with distances scaled in the units c/ω, the origin located at the ECR surface, the x - axis along the major radius R at the mid-plane, and (y, z) co-ordinates imitating poloidal and toroidal directions, respectively.
We consider waves approaching the EC resonance from the low - field side. Then the electrostatic approximation is valid for EBW with the dispersion

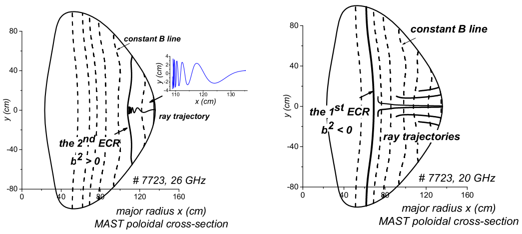

Figure 1: In the case of concave surface the rays oscillate regularly around the mid-plane (left figure). In the case of convex surface the rays deviate off the mid-plane exponentially (right figure).

relation $\varepsilon\left(n_{\|}, n_{\perp}\right)=0$ where the plasma dielectric function ε can be written

$$\varepsilon=1+\frac{2\upsilon}{n_{\perp}^{2}\beta^{2}}\left(1+e^{-\lambda}\sum_{n=-\infty}^{\infty}\frac{I_{n}(\lambda)Z(\xi_{n})}{\beta n_{\|}}\right). \tag{1}$$

Here $\upsilon=\omega_{pe}^{2}/\omega^{2}$, $\beta=\nu_{te}/c$, $\nu_{te}=\sqrt{2T_{e}/m_{e}}$, $\lambda=k_{\perp}^{2}\rho_{e}^{2}=\left(n_{\perp}q\beta\right)^{2}/2$, $q=\omega/\omega_{ce}\left(x,y\right)$, Z is the plasma dispersion function with the argument $\xi_{n}=(q-n)/q\beta n_{\|}$ and we assume $n_{\perp}\gg n_{\|}$. Consider the region $n_{\|}\beta\ll|q-s|/q\ll 1$ located close to the $q=s$ resonance but outside the resonance layer. Under this condition, $|\xi_{n}|\gg 1$ for all n and, as the first approximation, the Z functions in Eq.(1) can be replaced by the first term of their asymptotic expansion, $Z(\xi)\rightarrow-(1/\xi+1/2\xi^{3}+...)$ with the result $\varepsilon\left(n_{\perp},n_{\|}\right)\rightarrow\varepsilon_{0}$, $\varepsilon_{0}=\varepsilon\left(n_{\perp},0\right)$. This approximation is rather accurate in a plasma slab with $q=q(x)$. To take into account poloidal dependence of the magnetic field we have to calculate the resonance term $q=s$ in Eq.(1) with higher accuracy, adding second term of the Z asymptotic expansion and putting

$$\left(q\left(x,y\right)-s\right)/q\left(x,y\right)=\Delta(x)-y^{2}/b^{2}$$

where $\Delta(x)\rightarrow x/L$ at $x\ll L$, L is the scale-length of the magnetic field variation, y^{2}/b^{2} and $n_{\|}\beta$ are assumed small compared to Δ. Then $\varepsilon=\varepsilon_{0}+\varepsilon_{1}$,

$$\varepsilon_{0}=1+\frac{2\upsilon}{n_{x}^{2}\beta^{2}}\left(1-\frac{1}{\sqrt{\pi}sn_{x}\beta\Delta}\right),\ \varepsilon_{1}=-\frac{2\upsilon}{\sqrt{\pi}sn_{x0}^{3}\beta^{3}}\left(\frac{y^{2}}{\Delta^{2}b^{2}}+\frac{\beta^{2}n_{\|}^{2}}{2\Delta^{3}}\right).$$

The parallel index of refraction $n_{\|}=\boldsymbol{nB}/|\boldsymbol{B}|$ can be written close to the mid-plane as $n_{\|}=a\left(n_{y}+\gamma n_{z}-yn_{x0}/R_{f}\right)$ where $a=B_{p}/B$, $\gamma=B_{T}/B_{p}$ and R_{f} is the flux surface radius of curvature with B_{p} and B_{T} being the

poloidal and toroidal magnetic field components, respectively.
The function ε is the Hamiltonian function for the ray equations, but it is more convenient to present these equations in the form where the x coordinate is taken as the independent variable. To perform a transformation to this variable one has to find $n_x = N(x, y, n_y)$ from the dispersion relation $\varepsilon = 0$ and use N as a Hamiltonian function in canonical equations for y and n_y. Now, re-scaling the independent variable the ray equations can be written

$$\frac{dy}{d\tau} = -\frac{\partial K}{\partial n_y}, \frac{dn_y}{d\tau} = \frac{\partial K}{\partial y} \tag{2}$$

where $K = \delta^{1/2}(\tau)y^2/b^2 + \delta^{-1/2}(\tau)n_\parallel^2/2$ and $\delta = \Delta/\beta^2$. To get oriented note that at large τ, i.e. sufficiently close to the ECR surface, $\tau \sim L/\Delta^{5/6}$. It is clear that Eq.(2) describes an oscillatory motion. Assuming these oscillations adiabatic, find their frequency: $\Omega = a/\sqrt{2}b$. The adiabatic condition $\Omega\tau \gg 1$ takes the form $L/b\Delta^{5/6} \gg 1$ close to the resonance layer and is satisfied there if $b < L/n_\parallel\beta$. At $L \sim b$ the adiabatic condition is valid in the whole region of large $n_\perp$. The inequality $\Omega\tau \gg 1$ implies existence of the adiabatic invariant $I = K/\Omega$ that permits one to perform comprehensive analysis of the rays. In particular,

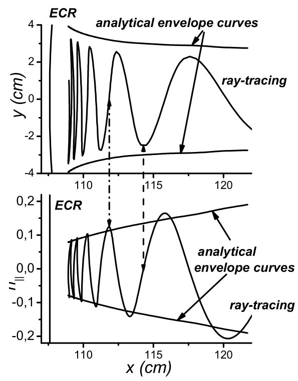

Figure 2: The curves $y(x)$ and $n_\parallel(x)$ from ray data and their envelopes. Y and $N_\parallel$ according to Eq.(3). The $\pi/2$ phase shift between the curves is indicated.

$$y = Y \sin\left(\int^\tau \Omega(t)dt + \alpha\right), n_\parallel = N_\parallel \cos\left(\int^\tau \Omega(t)dt + \alpha\right) \tag{3}$$

where $Y/b = (2\delta)^{-1/4}\sqrt{Ia/b}$, $N_\parallel = (2\delta)^{1/4}\sqrt{Ia/b}$ and α is a constant. Comparison of these expressions with ray data is presented in (fig.2).

The damping can be included into the ray equations (2) in a standard way by taking into account the imaginary part ε'' of the dielectric function treated as a perturbation. The damping rate is $\Gamma(\infty) \gg 1$.

3 Solution of the EBWs wave-equation

From the results of preceding Section one can draw a conclusion that in the case of concave ECR surface there is a sort of the "potential well" in the y direction, slowly varying with x. The EBWs must propagate in this plasma wave-guide as a discrete set of eigenmodes whose damping might be different from one predicted by the ray theory. To check this hypothesis, we now solve the EBW wave equation in the model applied above to the ray analysis. In the electrostatic approximation, the wave equation is the Poisson equation $\Delta\varphi + 4\pi\rho(\varphi) = 0$ where $\varphi(\boldsymbol{r})$ is the electrostatic potential and $\rho(\varphi)$ is the charge density, induced by the wave. In the linear approximation

$$4\pi\rho(\boldsymbol{r}) = \int g(\boldsymbol{r}, \boldsymbol{r}')\phi(\boldsymbol{r}')d\boldsymbol{r}' \tag{4}$$

and we assume the response function g in the form

$$g\left(\boldsymbol{R}, \boldsymbol{\rho}\right) = \frac{1}{\left(2\pi\right)^3}\int d\boldsymbol{n} g\left(\boldsymbol{R}, \boldsymbol{n}\right)\exp\left(-i\boldsymbol{n}\boldsymbol{\rho}\right) \tag{5}$$

where $\boldsymbol{R} = \left(\boldsymbol{r} + \boldsymbol{r}'\right)/2$, $\boldsymbol{\rho} = \boldsymbol{r} - \boldsymbol{r}'$ and $g\left(\boldsymbol{R}, \boldsymbol{n}\right) = n^2\left(1 - \varepsilon\left(\boldsymbol{R}, \boldsymbol{n}\right)\right)$ with $\varepsilon\left(\boldsymbol{R}, \boldsymbol{n}\right)$ given by Eq.(1) is the response function (in the wave vector representation) of homogeneous plasma with equilibrium parameters equal to their values in the real plasma at the spatial point $\boldsymbol{R}$. Assumed form of the response function $g\left(\boldsymbol{r}, \boldsymbol{r}'\right)$, natural for the case of a weakly inhomogeneous medium, it provides a symmetry with respect to the $\boldsymbol{r} \leftrightarrow \boldsymbol{r}'$ transform and guarantees physically meaningful results of the following calculations. In accordance with Equation for ϵ, $g\left(\boldsymbol{R}, \boldsymbol{\rho}\right) = g_0\left(\boldsymbol{R}, \boldsymbol{\rho}\right) + g_1\left(\boldsymbol{R}, \boldsymbol{\rho}\right)$ where $g_0\left(\boldsymbol{n}\right) = n^2\left(1 - \varepsilon_0\left(\boldsymbol{n}\right)\right),\, g_1\left(\boldsymbol{n}\right) = -n^2\varepsilon_1\left(\boldsymbol{n}\right),$ obtain

$$\Delta\varphi\left(\boldsymbol{r}\right) + \int g\left(\boldsymbol{R}, \boldsymbol{\rho}\right)\varphi\left(\boldsymbol{r}'\right)d\boldsymbol{r}' = 0. \tag{6}$$

The solution to this equation we will seek in the form

$$\varphi = \varphi_0(x)\psi(x,y), \varphi_0 = \frac{\exp\left(i\int^x n_{x0}(x')dx'\right)}{\sqrt{n_{x0}^2\partial\varepsilon_0'/\partial n_{x0}}} \tag{7}$$

where φ_0 is the WKB solution of the zero-order wave equation, i.e. Eq.(6) with $\varepsilon \to \varepsilon_0$. Substituting Eq.(7) into Eq.(6), obtain

$$i\frac{\partial \psi}{\partial x} + \left(\frac{\partial \varepsilon_0'}{\partial n_{x0}}\right)^{-1} \int_{-\infty}^{\infty} \varepsilon_1 \left(X, Y, \rho_x, \rho_y\right) \psi\left(x', y'\right) dx'dy' \qquad (8)$$

Straightforward although rather lengthy calculations lead from this equation to a partial differential equation for the function $f(x,y) = \psi(x,y)\exp\left(-i\gamma n_z y - i n_{x0} y^2/2R_f\right)$

$$i\frac{\partial f(\tau, y)}{\partial \tau} = \hat{K} f(\tau, y), \qquad (9)$$

where $\tau = \tau(x)$ is the "effective time" mentioned above and $\hat{K} = \delta(\tau)^{1/2} y^2/b^2 + a^2/\left(2\delta^{1/2}\right)\partial^2/\partial y^2$ is obtained from the "classical" Hamiltonian function of the ray equations (2) by substitution $n_\parallel \to -ia\partial/\partial y$. Similar to the "classical" case, τ dependence of $\hat{K}$ is slow. Accordingly, we solve Eq.(9) using the adiabatic approximation, i.e. treating τ in the right-hand side of the equation as a parameter. In this approximation the general solution to Eq.(9) can be written

$$f(x,y) = \frac{1}{\sqrt{y_0}} \sum_n C_n F_n(\tau, y) \exp\left(i\int^{\tau} K_n(\tau')d\tau'\right), \qquad (10)$$

where C_n are an arbitrary constant, $F_n(y) = U_n(y/y_0)$ with U_n being solutions to the equation

$$\frac{\partial^2 U_n}{\partial \varsigma^2} + \left(I_n - \varsigma^2\right) U_n = 0, \qquad (11)$$

satisfying $\int U_n(\zeta) U_m(\zeta) d\zeta = \delta_{mn}$, and $y_0 = (2\delta)^{-1/4}(ab)^{1/2}$ is the amplitude of the rays oscillations Y at $I = 1$. Equation (11) is the well-known quantum mechanical equation for the linear oscillator. Its solutions U_n vanishing at $|\tau| \to \infty$ exists at $I_n = 2n+1$, $n = 0, 1...$ Slowly varying functions of "time" τ K_n in Eq.(9) are given by $K_n = I_n\Omega$. Damping of the eigenmodes U_n is found by including ε'' into Eq.(9) and calculating an imaginary correction to the eigenvalue K_n using the perturbation theory. According to this result, there is, effectively, a minimal $n_\parallel \sim n_\parallel^{(0)} = N_\parallel|_{I=0}$. In the case its value exceeds β at $\delta \sim 1$, i.e. $b < a\beta^{-2}$, relativistic effects in wave propagation and damping become unimportant.

Solution (10) of the wave equation represents the EBW electrostatic potential as a set of eigenmodes confined in y and travelling in the x (radial) direction. The WKB approximation is applicable to the large-n eigenmodes only and n^{th} mode can be described by the rays (3) with $I = 2n+1$. The

WKB approximation is, however, inapplicable to few first terms in Eq.(10), therefore a part of the solution is beyond the ray method. This part is of special interest because it represents the least damped and the most deeply penetrating part of the EBW beam. Determination of the eigenmode spectrum for the solution (10) is a difficult problem requiring detail analysis of the wave behavior in the intermediate region between the vicinity and the adiabatic domain close to the ECR layer. The simplest model is an "initial" condition $\psi(x_b, y) = \psi_b(y)$ imposed at the boundary $x = x_b$ of the adiabatic region. Then

$$C_n = \int_{-\infty}^{\infty} \exp\left(-i\gamma n_z y - i n_{x0} y^2/2R_f\right) \psi_b(y) F_n(y) dy. \tag{12}$$

The factor $\exp\left(-i\gamma n_z y - i n_{x0} y^2/2R_f\right)$ is typically a rapidly oscillating function in the scale of y_0. Its presence in Eq.(12) diminishes small n coefficients shifting the n spectrum of the solution toward large n and thus increasing the wave damping. This result is, in fact, a manifestation of the uncertainty principle not taking into account automatically in the "classical" ray method.

4 Summary

1. The EBWs propagating close to the tokamak mid-plane and capable of penetrating deep into the plasma are considered.
2. Ray behavior in the mid-plane region depends on the ECR surface shape. In the case of concave ECR surface the rays oscillate around the mid-plane while for the convex ECR surface their deviation from the mid-plane growths exponentially.
3. In the case of oscillating rays the ray equation has an additional (adiabatic) integral of motion. The presence of this integral explains basic features of ray trajectories.
4. Conception of rays is incorrect for small $n_\parallel$ case. The relevant part of the beam is described by low-order eigenmodes of the wave equation found in the present work.
5. There is effectively a minimal value of $n_\parallel$. In case its value is sufficiently large compared to β, relativistic effects in wave propagation and damping can be ignored.

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